Unanimity rule on networks

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We present a model for innovation, evolution, and opinion dynamics whose spreading is dictated by a unanimity rule. The underlying structure is a directed network, the state of a node is either activated or inactivated. An inactivated node will change only if all of its incoming links come from nodes that are activated, while an activated node will remain activated forever. It is shown that a transition takes place depending on the initial condition of the problem. In particular, a critical number of initially activated nodes is necessary for the whole system to get activated in the long-time limit. The influence of the degree distribution of the nodes is naturally taken into account. For simple network topologies we solve the model analytically; the cases of random and small world are studied in detail. Applications for food-chain dynamics and viral marketing are discussed.

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I. INTRODUCTION

In general, a discovery, invention, or the emergence of something new depends on the combination of several parameters, all of them having to be simultaneously met. One may think of economy, where the production of a good depends on the production or existence of other goods, e.g., to produce a car one needs the wheel, the motor, and some fioritura. In return, this new discovery opens new possibilities and needs that will lead to the production of yet new goods, e.g., the simultaneous existence of the car and of alcohol directly leads to the invention of the air bag, etc. This feedback is responsible for the potential explosion of the number of items, such as observed, e.g., in the Cambrian explosion, and may even lead to a pandemics where all possible items are produced [1].

Such autocatalytic processes are very general [1–4] and obviously apply to many situations not only related to innovation, but also to evolution, opinion formation, food chains, etc. Typical examples are the dynamics of scientific ideas, music genres, or any other field where the emergence of a new element possibly leads to new combinations and new elements. In the case of social systems, where it is well known that the activation of an agent may require simultaneous exposure to multiple active neighbors [5,6], one may think of the spreading of information or rumors between social agents who propagate information only after verifying its validity among several sources, as well as other collective phenomena such as riots, stock market herds, etc.

After mapping the above catalytic reactions onto a network structure, where nodes represent items (agents) and directed links show which items are necessary for the production of others (which agents influence others), it is tempting to introduce a unanimity rule (UR): a node on the network is activated only if all the nodes arriving to it through a link are activated. Surprisingly, the dynamics of such an unanimity rule, that is a straightforward generalization of the majority rule of opinion dynamics [7–11] and reminds on features of the voter model [12–16], the Axelrod model [17,18] as well as of Boolean networks [3,19], is poorly known [1]. Let us emphasize that UR differs from these previous models by the fact that it is irreversible, i.e., once a node has reached the activated state, it remains in it. From a practical point of view, the irreversible nature of UR makes it an excellent candidate for modeling the adoption of a new technology, e.g., multimedia messaging service (MMS) [20], by interacting customers. Indeed, technological standards are themselves irreversible once they are adopted by a population, e.g., a mainstream revival of vinyl records instead of CD's and MP3's is more than unlikely. Another specificity of UR is the fact that it is purely deterministic, i.e., once the topology is fixed and an initial number of nodes are activated, the whole dynamics is determined by the interaction between neighbors. In contrast, the Voter model, when it is applied to a complex network, incorporates a random step when a node chooses among its neighbors with whom it will interact. Similarly, in the majority rule, a node choses randomly two nodes among its neighbors in order to form a majority triplet. We will show below that such random effects alter the spreading on the network and may lead to qualitatively very different features.

The unanimity rule may also be viewed as a limiting case of a threshold model (TM) for decision making scenarios [21–23], except that TM is usually applied to an undirected network while UR is defined on a directed network. In such a model, a node changes its state if a fraction T, $0 \le T \le 1$, of its neighbors are in the other state. However, contrary to previous studies, we are not interested in the probability that a cascade is triggered by a single node (or small set of nodes) nor in the expected size of the global cascade once it is triggered, but in the evolution of the system when a finite fraction of the nodes is initially activated. In the following, we will therefore look at the relation between the initial number of activated nodes and the final number of activated nodes in the network, and at the condition for a pandemics, i.e., a complete activation of the network, to take place.

The remainder of the paper is organized as follows. In Sec. II, the unanimity rule is introduced. In Sec. III, we derive equations for the time evolution of the proportion of

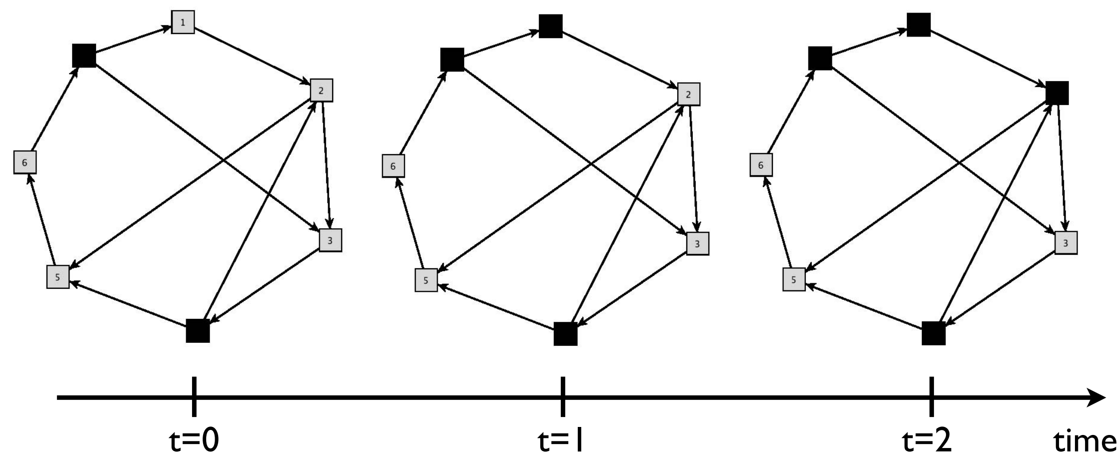

FIG. 1. First two steps of UR starting from an initial network of seven nodes, two of them being activated. Initially there is only one node among the nonactivated nodes that satisfies the unanimity rule. It gets therefore activated at the first time step. At that time, there is a new node whose two incoming links come from activated nodes. It gets activated at the second time step. It is straightforward to show that this system gets fully activated at the fourth time step.

activated nodes. These equations are shown to be nonlocal in time, i.e., they depend explicitly on the initial conditions. They depend on the network topology and exhibit a transition depending on the initial conditions, i.e., one needs to activate initially a minimum number of nodes to ensure that the whole system gets activated in the long-time limit. We also focus on a simplified topology where each node has exactly two incoming links and re-formalize our description in order to highlight the role played by the local correlations. To do so, we look at all the configurations of nodes and their direct neighbors and show that the system asymptotically reaches a frozen state. In Sec. IV, we successfully compare our predictions with simulations of UR on various topologies. In Sec. V we conclude and make some remarks on practical applications and generalizations of UR.

II. UNANIMITY RULE

Let us now introduce the model in detail. The network is composed of N nodes related through directed links. Each node exists in one of two states: activated or inactivated. The number of nodes with indegree i (the indegree of a node is defined to be the number of links pointing to it) is denoted by N_i and depends on the underlying network structure. It is therefore a fixed quantity that does not evolve with time. Initially (at $t=0$) there are $A(0)$ nodes which are activated, among which $A_i(0)$ have an indegree i. In general, the total number of activated nodes at time t is denoted by $A(t)$ and the number of nodes of type i, activated at time t, is $A_i(t)$. These quantities satisfy the relation

$$A(t) = \sum_i A_i(t). \tag{1}$$

It is also useful to introduce the quantities $n_i=N_i/N$ and $a_i(t)=A_i(t)/N_i$ which are the proportions of nodes with indegree i in the network (indegree distribution) and the probability that such a node i is activated, respectively. Let us also define

$$a(t) = \frac{A(t)}{N} = \sum_i n_i a_i(t) \tag{2}$$

that is the fraction of activated nodes in the whole network at time t.

The unanimity rule is defined as follows (see Fig. 1). At each time step, each node is considered, i.e., the dynamics is synchronous [24]. If all the links arriving to an inactivated node i originate at nodes which are activated at $t-1$, i gets activated at t. Otherwise, it remains inactivated. The process is applied iteratively until the system reaches a stationary frozen state, characterized by an asymptotic value $a_{\text{FIN}} \equiv a(\infty)$. In the following, we are interested in the relation between a_{FIN} and $a_{\text{IN}} \equiv a(0)$, i.e., what is the final occupation of the network as a function of its initial occupation on a specific network. Let us mention that each node may be produced by only one combination of (potentially many, depending on the indegree) nodes. This is a modification of the model of Hanel *et al.* [1], where more than one pairs of (two) nodes could produce new elements and will lead to a different equation for the activation evolution, as shown below. The dynamics studied here implies that nodes with a higher indegree will be activated with a probability smaller than those with a smaller indegree—because the former have more conditions to be fulfilled.

III. ANALYTICAL APPROACH

A. Equation of evolution

Let us now derive an equation of evolution for $A_i(t)$ and $A(t)$. To do so it is helpful to consider the first time step and then to iterate. There are initially $A(0)$ activated nodes,

$A_i(0)=A(0)N_i/N$ of them being of indegree i on average (the activated nodes are randomly chosen in the beginning). The ensemble of $A_i(0)$ nodes is called the initial set of indegree i. By construction, the probability that i randomly chosen nodes are initially activated, is $a^i(0)$ (i is an exponent). The average number of nodes with indegree i and which respect the unanimity rule is therefore $N_i a^i(0)$. Among these nodes, $N_i a(0) a^i(0)$ were already activated initially. This is due to the fact that the total number of nodes with degree i that are initially activated is $N_i a(0)$. Consequently, the number of nodes that gets activated at the first time step is

$$\Delta_i(0) = [N_i - N_i a(0)] a^i(0) \tag{3}$$

and, on average, the total number of occupied nodes with indegree i evolves as

$$A_i(1) = A_i(0) + \Delta_i(0). \tag{4}$$

At the next time step, the average number of nodes with indegree i, which respect the unanimity rule and which are outside the initial set is $[N_i - N_i a(0)] a^i(1)$. Among those nodes, $\Delta_i(0)$ have already been activated during the first time step, so that the average number of nodes which get activated at the second time step is

$$\Delta_i(1) = [N_i - N_i a(0)][a^i(1) - a^i(0)]. \tag{5}$$

Note that Eq. (5) is valid because no node in $\Delta_i(1)$ also belongs to $\Delta_i(0)$. This is due to the fact that each node can only be activated by one combination of i nodes in our model, so that no redundancy is possible between $\Delta_i(1)$ and $\Delta_i(0)$. By proceeding similarly, it is straightforward to show that the contributions $\Delta_i(t)$ read

$$\Delta_i(t) = [N_i - N_i a(0)][a^i(t) - a^i(t-1)], \tag{6}$$

with $a(-1)=0$, by convention. The number of activated nodes evolve as

$$A_i(t+1) = A_i(t) + \Delta_i(t). \tag{7}$$

By dividing Eq. (7) by N_i, one gets a set of equations for the proportion of nodes $a_i \in [0,1]$:

$$a_i(t+1) = a_i(t) + [1 - a(0)][a^i(t) - a^i(t-1)], \tag{8}$$

where the coupling between the different proportions $a_i(t)$ occurs through the average value $a(t)=\Sigma_i n_i a_i(t)$, as defined above. Finally, by multiplying Eq. (8) by the indegree distribution n_i and summing over all values of i, one gets a closed equation for the average proportion of activated nodes in the network that reads

$$a(t+1) = a(t) + [1 - a(0)] \sum_i n_i [a^i(t) - a^i(t-1)]. \tag{9}$$

Let us stress that Eq. (9) is nonlinear as soon as $n_i \neq 0$, $i > 1$. Moreover, it is characterized by the nontrivial presence of the initial condition $a(0)$ in the right-hand side nonlinear term and is therefore nonlocal in time. One should stress that this nonlocality is a feature of the effective mean field description and not of the UR itself, where, by construction, the configuration of the system at time $t+1$ is fully determined by its configuration at time t. The origin for this nonlocality in the mean field description will be discussed further in Sec. III C. Finally, let us also note that Eq. (9) explicitly shows how the indegree distribution n_i affects the propagation of activated nodes in the system.

B. Some special cases

Let us now focus on simple choices of n_i in order to apprehend analytically the behavior of Eq. (9). The simplest case is $n_i=\delta_{i1}$ (each node has one incoming link) for which Eq. (9) reads

$$a(t+1) = a(t) + [1 - a(0)][a(t) - a(t-1)]. \tag{10}$$

This equation is solved by recurrence:

$$a(1) = a(0) + [1 - a(0)] a(0),$$

$$\begin{aligned} a(2) &= a(0) + [1 - a(0)] a(0) \\ &\quad + [1 - a(0)]\{a(0) + [1 - a(0)] a(0) - a(0)\} \\ &= a(0) + [1 - a(0)] a(0) + [1 - a(0)]^2 a(0) \end{aligned} \tag{11}$$

and, in general,

$$a(t) = \sum_{u=0}^{t} [1 - a(0)]^u a(0) = 1 - [1 - a(0)]^{t+1}. \tag{12}$$

This last expression is easily verified:

$$\begin{aligned} a(t+1) &= \sum_{u=0}^{t} [1 - a(0)]^u a(0) + [1 - a(0)] \\ &\quad \times \left(\sum_{u=0}^{t} [1 - a(0)]^u a(0) - \sum_{u=0}^{t-1} [1 - a(0)]^u a(0) \right) \\ &= \sum_{u=0}^{t+1} [1 - a(0)]^u a(0). \end{aligned} \tag{13}$$

The above solution implies that any initial condition $a(0) \neq 0$ converges toward the asymptotic state $a_{\mathrm{FIN}}=1$, i.e., whatever the initial condition, the system is fully activated in the long time limit. From Eq. (12), one finds that the relaxation to $a_{\mathrm{FIN}}=1$ is exponentially fast, $a(t) \approx 1 - e^{t \ln[1-a(0)]}$.

Let us now focus on the more challenging case $n_i=\delta_{i2}$ where all the nodes have an indegree of 2 by construction. In that case, Eq. (9) reads

$$a(t+1) = a(t) + [1 - a(0)][a^2(t) - a^2(t-1)]. \tag{14}$$

The nonlinear term does not allow one to find a simple recurrence expression as above. However, numerical integration of Eq. (14) shows that the leading terms in the Taylor expansion of $a(t)$ behave as

$$a(t) = \sum_{i=1}^{t+1} a^i(0) + O(t+2), \tag{15}$$

thus suggesting that the asymptotic solution is

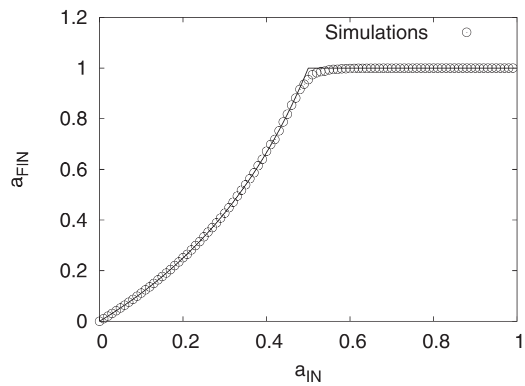

FIG. 2. Relation $a_{\mathrm{FIN}}(a_{\mathrm{IN}})$ obtained by integrating numerically Eq. (14) (solid line) and by performing simulations of the model on a network with $n_i = \delta_{i2}$. The system obviously shows a transition at $a(0)=1/2$. Moreover, the prediction, Eq. (16) is in perfect agreement (indistinguishable from the solid line) with the numerical integration of Eq. (14). The simulations were performed with $N = 10\,000$ nodes and the results are averaged over 100 realizations of the process.

$$a(\infty) = \frac{a(0)}{1 - a(0)}. \tag{16}$$

This solution should satisfy the normalization constraint $a(\infty) \leq 1$, so that it can hold only for initial conditions $a(0) < 1/2$. This argument suggest that a transition takes place at $a_c = 1/2$, such that only a fraction of the whole system gets activated when $a(0) < a_c$ while the whole system activates above this value (see Fig. 2). We verify the approximate solution (16) by looking for a solution of the form

$$a(t) = \frac{a(0)}{1 - a(0)}[1 + \epsilon(t)]. \tag{17}$$

By inserting this expression into Eq. (14), one gets the recurrence relations

$$\epsilon(t+1) = \epsilon(t) + a(0)[1 + \epsilon(t)]^2 - a(0)[1 + \epsilon(t-1)]^2,$$

$$\epsilon(t+1) = \epsilon(t) + 2a(0)[\epsilon(t) - \epsilon(t-1)], \tag{18}$$

where the second line is obtained by keeping only first order corrections in ϵ. In the continuous time limit, keeping terms until the second time derivative, one obtains

$$[1 - 2a(0)]\partial_t \epsilon(t) + 1/2[1 + 2a(0)]\partial_t^2 \epsilon(t) = 0, \tag{19}$$

whose exponential solutions read $\epsilon(t) = e^{-\lambda t}$ with

$$\lambda = \frac{1}{2}\frac{[1 - 2a(0)]}{[1 + 2a(0)]}. \tag{20}$$

This is a relaxation to the stationary state $a(\infty)$, only when $a(0) < 1/2$, thereby confirming a qualitative change at $a_c = 1/2$. Contrary to the case δ_{i1}, there is therefore a transition, reminding the behavior observed in Refs. [1,22] and the existence of critical mass, i.e., a critical value $a_c = 1/2$ under which only a fraction of the whole system is asymptotically activated.

Before going further, it is interesting to focus on a variant of UR in order to highlight the importance of its deterministic nature. This variant is defined as follows. Let us consider a directed network where nodes have on average a high indegree. For the sake of simplicity, we consider a fully connected network, i.e., each node receives a link from all the $N-1$ other nodes. At each time step, all the nodes randomly select two of their neighbors, in a way that reminds on the process of Eq. (14) and it gets activated only if both neighbors were activated at the previous time step. It is straightforward to show that the equation of evolution for a_t is now

$$a(t+1) = a(t) + [1 - a(t)]a^2(t) \tag{21}$$

and that $a_{\mathrm{FIN}} = 1$, i.e., all the nodes are asymptotically activated, whatever the initial condition a_{IN}, except if $a_{\mathrm{IN}} = 0$. The difference from the relation Eq. (16) of UR is due to the fact that the inclusion of random effects mixes the different configurations of the system, i.e., in a mean field way, and therefore multiplies the possibility for nodes to be activated.

C. Alternative description: The role of correlations

Let us now return to UR and emphasize some points that deserve attention. First, one should note that the critical parameter of the above transition is not an external parameter, but it is the initial condition a_{IN} itself, i.e.,

$$a_{\mathrm{FIN}} \begin{cases} = 1 & \text{if } a_{\mathrm{IN}} > a_c, \\ < 1 & \text{otherwise.} \end{cases} \tag{22}$$

Such a dependence on initial conditions has also been observed in Axerod dynamics [25–27] or minority games [28]. Equation (22) also implies that UR has a continuum of attractors, i.e., the asymptotic state of the systems is not limited to a few fixed points, each of them surrounded by its own basin of attraction, but the whole range of values $a \in [0,1]$ may be a stationary solution depending on the initial condition a_{IN} [this can be seen from Fig. 2 as the curve $a_{\mathrm{FIN}}(a_{\mathrm{IN}})$ goes continuously from 0 to 1]. Finally, one should also stress that the nonlocality in time of the dynamical equations Eq. (9) implies that the same value $a(t)$ will reach a different stationary state a_{FIN} depending on the time t at which it is attained.

In order to understand the origin of these peculiar properties, it is useful to tackle the problem analytically from a different point of view. To do so, let us focus on networks where all the nodes have an indegree of 2, i.e., $n_i = \delta_{i2}$. The state of a node may either be A for activated or I for inactivated, but, in order to calculate its state at the next step, one also needs to know the state of its two neighbors. Consequently, we represent the state of a node by a triangle (see Fig. 3) composed of this node and of its two incoming neighbors. Let $N_{\alpha_0;\alpha_1\alpha_2}$ be the number of such triangular configurations where a node in state α_0 receives its first link from a node in state α_1 and a second link from a node in state α_2. α_i may be A (activated) or I (inactivated).

The equations of evolution for $N_{\alpha_0;\alpha_1\alpha_2}$ are easily found to be (see Fig. 3)

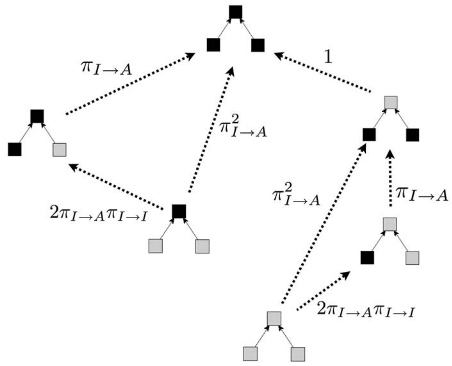

FIG. 3. Transition probabilities between the six possible triangular configurations. The dynamics is obviously irreversible, with a preferred direction toward the activated (black) state.

$$N_{A;AA}(t+1) = N_{A;AA}(t) + N_{I;AA}(t) + \pi_{I\to A}N_{A;AI}(t) + \pi^2_{I\to A}N_{A;II}(t),$$

$$N_{A;II}(t+1) = \pi^2_{I\to I}N_{A;II}(t),$$

$$N_{A;AI}(t+1) = \pi_{I\to I}N_{A;AI}(t) + 2\pi_{I\to A}\pi_{I\to I}N_{A;II}(t),$$

$$N_{I;II}(t+1) = \pi^2_{I\to I}N_{I;II}(t),$$

$$N_{I;AA}(t+1) = \pi^2_{I\to A}N_{I;II}(t) + \pi_{I\to A}N_{I;AI}(t),$$

$$N_{I;AI}(t+1) = \pi_{I\to I}N_{I;AI}(t) + 2\pi_{I\to A}\pi_{I\to I}N_{I;II}(t), \quad (23)$$

where we have taken into account the fact that an inactivated node whose two incoming neighbors are activated will be activated at the next time step and that an activated node remains activated forever. The quantities $\pi_{I\to A}$ and $\pi_{I\to A}$ are the probabilities for an inactivated incoming neighbor to get activated and to remain inactivated, respectively. These quantities obviously respect

$$\pi_{I\to I} = 1 - \pi_{I\to A}. \quad (24)$$

In order to close the system of equations (23), we evaluate the transition probability $\pi_{I\to A}$ by the probability for a randomly selected node I to have two activated incoming neighbors

$$\pi_{I\to A} = \frac{N_{I;AA}}{N_{I;AA} + N_{I;AI} + N_{I;II}}. \quad (25)$$

Let us also introduce the total number of activated nodes $N_A = \Sigma_{\alpha_1,\alpha_2}N_{A;\alpha_1\alpha_2}$ [which is equal to A of Eq. (1)] and the total number of inactivated nodes $N_I = \Sigma_{\alpha_1,\alpha_2}N_{I;\alpha_1\alpha_2}$. By summing over the states of the incoming neighbors, one also finds

$$N_A(t+1) = N_A(t) + N_{I;AA}(t),$$

$$N_I(t+1) = N_I(t) - N_{I;AA}(t), \quad (26)$$

which confirms that only the configurations $N_{I;AA}$ drive the evolution of the system and that a stationary state is reached when $N_{I;AA}=0$. This also shows that the stationary state is frozen, as no change is possible when $N_{I;AA}=0$. In some sense, the number of dynamic triangles $N_{I;AA}$ may therefore be viewed as the potential of the network to change, and the evolution stops, whatever its state, when all the dynamic triangles have been transformed into other triangles.

Let us now focus on the initial conditions of the system of equations (23). In principle, many initial conditions may be chosen, each of them leading to a different trajectory in the six-dimensional dynamical space. However, initial conditions are subject to several constraints. On the one hand, the following equality has to hold:

$$\sum_{\alpha_0,\alpha_1,\alpha_2} N_{\alpha_0;\alpha_1\alpha_2} = N, \quad (27)$$

which is just a normalization, but initial conditions must also satisfy the conservation law

$$T_A = 2N_A,$$

$$T_I = 2N_I, \quad (28)$$

where the quantities

$$T_A = 2N_{A;AA} + 2N_{I;AA} + N_{A;AI} + N_{A;AI},$$

$$T_I = 2N_{A;II} + 2N_{I;II} + N_{A;AI} + N_{A;AI} \quad (29)$$

are the total number of activated (inactivated) incoming neighbors in triangles. Relation (28) therefore means that each node that is an incoming neighbor in a triangle is also at the summit of another triangle (as it also receives two incoming links by construction). It is important to stress that the constraints (28) are preserved by the dynamics (23). Indeed, one verifies easily that

$$T_A(t+1) - 2N_A(t+1) = \pi_{I\to I}[T_A(t) - 2N_A(t)],$$

$$T_I(t+1) - 2N_I(t+1) = \pi_{I\to I}[T_I(t) - 2N_I(t)] \quad (30)$$

so that Eq. (28) holds all times if it holds at $t=0$.

One should emphasize that there are many configurations $N_{\alpha_0;\alpha_1\alpha_2}$ of the six-dimensional space that satisfy the constraints (27) and (28) and that have the same average number of activated nodes N_A. However, the $N_A = Na_0$ nodes that are initially activated are randomly chosen. Consequently, there are no correlations between the states of neighboring nodes and, among all the possible configurations for which $N_A = Na_0$, the initial condition is actually

$$N_{A;AA}(0) = Na^3(0),$$

$$N_{A;II}(0) = Na(0)[1 - a(0)]^2,$$

$$N_{A;AI}(0) = 2Na^2(0)[1 - a(0)],$$

$$N_{I;II}(0) = N[1 - a(0)]^3,$$

$$N_{I;AA}(0) = N[1 - a(0)]a^2(0),$$

$$N_{I;AI}(0) = 2N[1 - a(0)]^2 a(0). \tag{31}$$

One verifies easily that Eq. (31) respects the constraints (27) and (28).

A recursive integration of the system of Eqs. (23) starting from the initial conditions (31) has been performed by using MATHEMATICA. It is found that the resulting $a(t)$ is identical to that obtained by integrating Eq. (14) starting from the same initial condition $a(0)$, so that the relation $a_{\mathrm{FIN}}(a_{\mathrm{IN}})$ is also identical. However, an analytical demonstration of the equivalence of Eqs. (23) and (14) is still lacking—we are open to suggestions.

Looking at Eqs. (23) and (14) as two sides of the same process, one may now understand why the same value of $a(t)$ may lead to different stationary solutions a_{FIN}. Indeed, it is easy to show that the system of Eqs. (23) develops correlations in the course of time, i.e., in general, the configurations $N_{\alpha_0;\alpha_1\alpha_2}(t)$ cease to fulfill the following relations when $t>0$:

$$N_{A;AA}(t) = Na^3(t),$$

$$N_{A;II}(t) = Na(t)[1 - a(t)]^2,$$

$$N_{A;AI}(t) = 2Na^2(t)[1 - a(t)],$$

$$N_{I;II}(t) = N[1 - a(t)]^3,$$

$$N_{I;AA}(t) = N[1 - a(t)]a^2(t),$$

$$N_{I;AI}(t) = 2N[1 - a(t)]^2 a(t), \tag{32}$$

where $a(t)=N_A/N$. This result is obvious for $t\to\infty$ and $0<a_{IN}<a_c$, as we have shown that $0<a(\infty)<1$, while $N_{I,AA}=0$ in that case. The emergence of such correlations implies that the same value of $a(t)$, for different values of t, may correspond to different configurations $N_{\alpha_0;\alpha_1\alpha_2}$ and may therefore lead to a different trajectory in the six-dimensional space. Consequently, a different asymptotic state a_{FIN} may be reached in principle.

Before going further, one should note that a generalization of Eqs. (23) for more general degree distributions n_i is not an easy task, as it implies a multiplication of the number of variables to take into account. The formalism (9) is to be preferred in that case.

IV. SIMULATIONS

Let us now verify the above predictions by performing numerical simulations of the model. To do so, one has first to build networks whose indegrees are δ_{iK}, i.e., the indegree of each node is exactly K, where $K=1$ for Eq. (12) and $K=2$ for Eq. (16). Such networks are easily implemented by picking randomly K nodes $l_1,l_2,\ldots,l_K$ for each node $i\in[1,N]$ and adding links going from $l_1,l_2,\ldots,l_K$ to i. Once the underlying network is built, we randomly assign $a(0)N$ activated nodes to the network and apply the unanimity rule until a stationary state is reached. The asymptotic value $a_{\mathrm{FIN}}\equiv a(\infty)$ is averaged over several realizations of the process (on several realizations of the underlying network) and is shown to be in excellent agreement with the theoretical predictions. The case $K=2$ is plotted in Fig. 2, but other values of K have also been studied and suggest that $a_c(K)=1-1/K$. This behavior is expected as nodes with a higher degree require more conditions for an activation, so that the asymptotic number of activated nodes is reduced.

We have also focused on more realistic topologies and compared the results obtained from Eq. (9) with numerical simulations of the UR. Two types of networks are discussed in the following, purely random networks [29] and small-world-like networks [30], but other types have also been considered and lead to the same conclusions. The random network was obtained by randomly assigning L directed links over N nodes, so that its degree distribution is

$$n_k = e^{-\lambda}\frac{\lambda^k}{k!}, \tag{33}$$

where $\lambda=L/N$. The small-world network was obtained by starting from a directed ring configuration and then randomly assigning L directed links (shortcuts) over the nodes, i.e., the total number of links in that case is $L+N$ (The network drawn in Fig. 1 is such network with $N=7$ nodes and $L=3$ short cuts). In that case, the degree distribution is easily found to be

$$n_k = \begin{cases} 0 & \text{if } k=0, \\ e^{-\lambda}\dfrac{\lambda^{k-1}}{(k-1)!} & \text{otherwise.} \end{cases} \tag{34}$$

Let us note that the directed small-world network can be viewed as a food chain with a well-defined hierarchy between species together with some random short cuts. In that case, UR can be interpreted as an extinction model: if *all* the species that one particular species can eat, go extinct, this species will also die out.

To compare the simulation results with the theoretical predictions, we integrate Eq. (9) with the corresponding degree distributions of Eqs. (33) and (34). The agreement is excellent, except close to the transition points where finite size effects are expected. One observes [Fig. 4(b)] by increasing the average degree that the location of the transition a_c is closer and closer to 1, for the same reason such a shift took place in the case of δ_{iK} networks.

One should also stress that each node receives at least one incoming link by construction ($n_0=0$) in δ_{iK} networks and in small-world network. This is not the case for random networks [see Eq. (33)], for which one has to discuss the ambiguous dynamics of nodes with zero incoming links. Two choices are possible. Either these nodes cannot be activated in the course of time, because they are not reached by any other node (no-zero version), or all of them get activated at the first time step, thereby assuming that their activation does not require any first knowledge (Zero version). The choice is a question of interpretation. The two versions are associated to different evolution equations, namely,

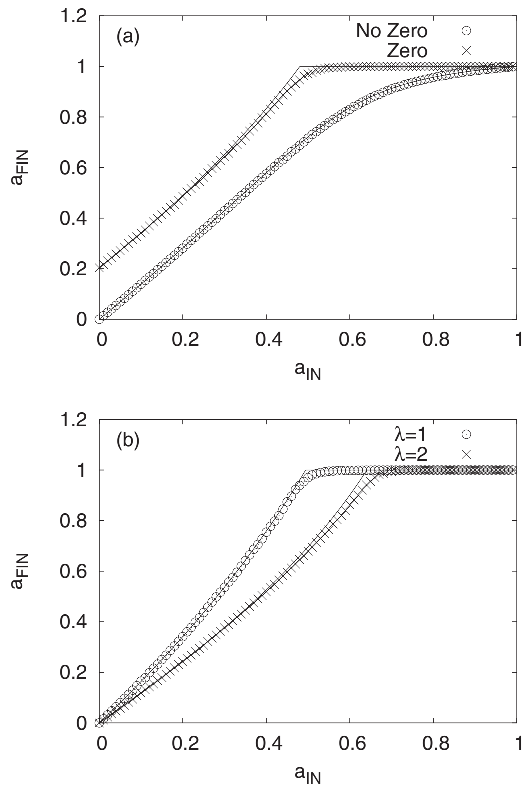

FIG. 4. (a) Relation $a_{FIN}(a_{IN})$ obtained for random networks with $L=2N$. The no zero and zero versions (see text) are shown. (b) $a_{FIN}(a_{IN})$ for small-world networks with $L=\lambda N$ short-cuts. The simulations were performed with $N=10\,000$ nodes and the results are averaged over 100 realizations of the process. The solid lines are the numerical solutions of Eqs. (35) and (36) and Eq. (9), respectively, evaluated with the corresponding degree distributions, given in Eqs. (33) and (34).

$$a(t+1) = a(t) + [1 - a(0)]\sum_{i=1}^{\infty} n_i[a^i(t) - a^i(t-1)] \quad (35)$$

for the no-zero version, and

$$a(t+1) = a(t) + [1 - a(0)]\sum_{i=0}^{\infty} n_i[a^i(t) - a^i(t-1)] \quad (36)$$

for the zero version. When $n_0=0$, the above equations are obviously equivalent. The two interpretations lead to quite different behaviors [Fig. 4(a)]. As expected, there are always more activated nodes in the zero version than in the no-zero version. This effect even provokes qualitative differences between the versions, i.e., as shown in Fig. 4(a), there is no critical value a_c for the no-zero version while $a_c \approx 0.48$ for the zero version.

Before concluding, let us stress that the difference between the two interpretations is more pronounced when the number of nodes with zero incoming links becomes higher. This is the case for growing networks, e.g., the Barabási-Albert network [31], that are well-known models for scale-free networks. We have verified this effect by studying numerically UR on a network that was built starting from one seed node and adding nodes one at a time until the system is composed of N nodes [32]. At each step, the node first connects to a randomly chosen node and, with probability r, it redirects its link to the father of the selected node. This method is well known to be equivalent to preferential attachment and to lead to the formation of fat-tail degree distributions $k^{-\nu}$, with $\nu=1+1/r$ [32,33], while the number of nodes with zero incoming links is very large. We have studied several values of ν close to $\nu=3$ and it is shown (see Fig. 5) that all the nodes are finally activated whatever the initial condition in the zero version ($a_c=0$), while there is no transition in the no zero version ($a_c=1$). It is also interesting to point that an integration of the Eqs. (35) and (36) reproduce the same extreme behavior.

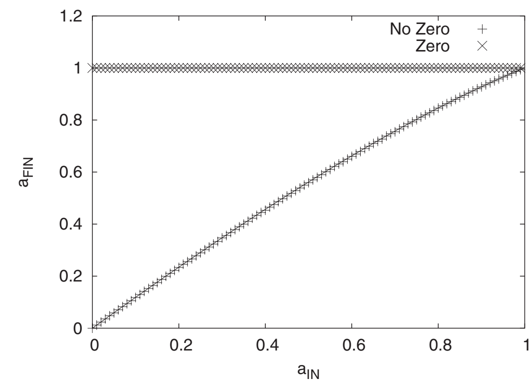

FIG. 5. $a_{FIN}(a_{IN})$ for a network with redirection. The total number of nodes is $N=10\,000$, $r=1/2$ and the results are averaged over 100 realizations. In the zero version, one observes that all the nodes are finally activated whatever the initial condition. In the no zero version, in contrast, the full activation of the network is attained only when $a_{IN}=1$. The solid lines correspond to numerical integrations of Eqs. (35) and (36) with the corresponding degree distribution.

V. CONCLUSION

To summarize, we have introduced a simple model for innovation whose dynamics is based on the unanimity rule. It is shown that the discovery of new items on the underlying network opens perspectives for the discovery of yet new items. This feedback effect may lead to complex spreading properties, embodied by the existence of a critical size for the initial activation, that is necessary for the complete activation of the network in the long-time limit. The problem has been analyzed numerically on a large variety of network structures and has been successfully described by recurrence relations for the average activation. Let us stress that these recurrence relations have a quite atypical form due to their explicit dependence on initial conditions. Moreover, their nonlinearity makes them a hard problem to solve in general. We have also shown that the system might be studied alternatively by focusing on the configurations of nodes and their direct neighbors, thereby highlighting the role of internal correlations and clarifying the origin of the nonlocality in time of the recurrence relations.

Finally, let us emphasize that unanimity rule is a general mechanism that should apply to numerous situations related to innovation, opinion dynamics, or even species and population dynamics. Practically, one may think of the adoption of a new technological standard in a population of interacting customers or the propagation of rumors between social agents, the key ingredient of UR being that many conditions have to be simultaneously met in order to drive the activation of a node. We have shown that UR naturally leads to the notion of critical mass, which might have important consequences, in marketing, for instance, as it suggests that an efficient viral marketing campaign [34] should reach a minimum number of customers in order to ensure the propagation of the message through the whole network. Moreover, the results of Sec. IV also suggest that a targeted attack [35,36], i.e., a strategic choice of initially activated nodes instead of a random choice, might alter, and possibly accelerate, the spreading of the process. This is due to the fact that many triangle configurations $N_{\alpha_0,\alpha_1,\alpha_2}$ correspond to the same value of $a(0)$, each of them corresponding to a different time evolution of $a(t)$.

Finally, one should also stress that UR is a very extreme dynamics that may lead to counterintuitive features, i.e., the propagation becomes slower as the network gets more connected. This effect can be circumvented by softening the unanimity rule, for instance, by requiring that only a finite number of neighbors has to be activated for an activation. We will show elsewhere that this variation—unfortunately more complicated—leads to qualitatively similar results (existence of a_c) without such unrealistic features.

To conclude, we hope that in the above sense this paper will form part of a set of works (activated nodes) which allow for the activation of novel (yet inactive) perspectives and research directions.

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